Solutions to Weekend Activity: Sigma Notation and Series

3) Find s_1, s_2, s_3, s_4, and s_5; Do not simplify your answers.

$s_1 = 1$
$s_2 = 1 + \frac{1}{4}$
$s_3 = 1 + \frac{1}{4} + \frac{1}{9}$
$s_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$
$s_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$

4) Note that s_n an increasing sequence (each term is bigger than the previous term). Why should we expect this to occur? Will this be true of the partial sums $s_n = \displaystyle\sum_{k=1}^{n} a_k$ for ANY sequence a_n?

The terms of the sequence of partial sums are increasing because $\frac{1}{n^2} > 0$ for all $n \in \mathbb{N}$. Therefore, we are always adding another positive fraction to the preexisting sum. This will not be true for s_n associated to a_n where a_n is ever less than or equal to zero.

So although the terms of $a_n = \frac{1}{n^2}$ get smaller and smaller, the terms of s_n keep getting get bigger and bigger. It's clear that $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$, but what is $\displaystyle\lim_{n\to\infty} s_n$?
*Unfortunately, **it could be just about anything!***

5) Explain why the limit of s_n is so unpredictable. Keep in mind your knowledge $0 \cdot \infty$.

As n increases, the terms of a_n get smaller and smaller, tending toward zero. However, the value of s_n keeps increasing, since we are adding together more and more terms. The size of the terms going to zero and the number of terms going to infinity means we have no idea what will happen: $0 * \infty$ is not well defined, and limits of type $0 * \infty$ are unpredictable. The answer will depend on the relative strengths of the 0 and ∞ involved. Generally, such limits must be solved with L'Hospitals. In this case, the question is whether the terms of a_n go toward zero **fast enough** for their sum to be finite.

6) Using your calculator, approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *by finding* s_{10} *as a decimal.*

$s_{10} \approx 1.54977$

7) The actual value of this sum is, surprisingly, $\pi^2/6$*. Estimate* $\pi^2/6$ *using your calculator, and compare the results to your approximation. Which is bigger? How would you know which is bigger without even checking the numbers?*

Using the approximation $\pi \approx 3.1415$, you get $\frac{\pi^2}{6} \approx 1.64484$
The value of $\frac{\pi^2}{6}$ should be bigger, since it is obtained from s_{10} by adding on still more positive fractions.

We say ***a series converges if and only if its sequence of partial sums converges.*** *Another way to say this would be* $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} s_k$

8) Use the definition of convergence to show that the series $\displaystyle\sum_{k=1}^{\infty} 0$ *converges to zero. Begin by finding a formula for the sequence of partial sums* s_n*.*

$s_n = \sum_{k=1}^{n} 0 = 0$. Therefore, since $\displaystyle\lim_{n\to\infty} s_n = 0$, we conclude that $\displaystyle\sum_{k=1}^{\infty} 0$ converges to zero by definition.

9) Use the definition of convergence to show that the series $\displaystyle\sum_{k=1}^{\infty} c$ *diverges for any constant* $c \neq 0$*. Begin by finding a formula for the sequence of partial sums* s_n*.*

$s_n = \sum_{k=1}^{n} c = c * n$. Therefore, $\displaystyle\lim_{n\to\infty} s_n$ diverges. We conclude that $\sum_{n=1}^{\infty} c$ diverges.

10) If we know that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges, what (if anything) can we say about*

a) $\displaystyle\lim_{n\to\infty} s_n$? The limit must converge

b) $\displaystyle\lim_{n\to\infty} a_n$? The limit must be zero

If $\displaystyle\sum_{k=1}^{\infty} a_k = A$ *and* $\displaystyle\sum_{k=1}^{\infty} b_k = B$ *(both converge to constants), what will* $\displaystyle\sum_{k=1}^{\infty}(2a_n + b_n)$ *do?*

Let $A_n = \displaystyle\sum_{k=1}^{n} a_k$, $B_n = \displaystyle\sum_{k=1}^{n} b_k$, and $s_n = \displaystyle\sum_{k=1}^{n}(2a_k + b_k)$ be the sequences of partial sums. The information given in the problem tells us that $\lim\limits_{n\to\infty} A_n = A$ and $\lim\limits_{n\to\infty} B_n = B$.
Note that, by the properties of sums, $s_n = 2A_n + B_n$ for all n.
So, by the properties of limits: $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} 2A_n + B_n = 2\lim\limits_{n\to\infty} A_n + \lim\limits_{n\to\infty} B_n = 2A + B$

11) I have told you that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. *Note that* $\frac{1}{n^3} \le \frac{1}{n^2}$ *for all* $n \ge 1$.
Use these facts to argue that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ *converges.*

Since $\frac{1}{k^3} \le \frac{1}{k^2}$, it follows that $\displaystyle\sum_{k=1}^{n} \frac{1}{k^3} \le \sum_{k=1}^{n} \frac{1}{k^2}$ Since the sequence of partial sums of $\frac{1}{k^3}$ is increasing and bounded above by $\pi^2/6$, it must converge.

12) Now, use the fact that $\int_1^{\infty} x^{-3}dx$ *converges to argue that* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ *converges.*
Hint: Draw a picture of $\int_1^{\infty} x^{-3}dx$ *being underestimated by rectangles of width 1.*

Each rectangle you have drawn has area $1/n^3$ where n is the right endpoint of each interval $[n-1, n], n \ge 2$. The sequence $\displaystyle\sum_{k=1}^{n} 1/k^3 = 1 + \sum_{k=2}^{n} 1/k^3$ is increasing and bounded above by the integral $1 + \int_1^n \frac{1}{x^3}dx$. Since this improper integral converges by the p-test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges as well.

13) Let $a_n = (-1)^{n+1}\frac{1}{n^3}$*. Consider* $\sum_{n=1}^{\infty} a_n$*. Note that* $s_5 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125}$*.* $\sum_{n=1}^{\infty} a_n$ *is called an alternating series since the terms of the sum alternate between being positive and negative. Does* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ *converge or diverge?*

$0 \leq \sum_{k=1}^{n}(-1)^{n+1}\frac{1}{n^3} \leq \sum_{k=1}^{n}\frac{1}{n^3}$, so the sum is bounded and cannot diverge to infinity. The oscillations of sequence of partial sums grow smaller and smaller as n increases. Therefore, the sequence of partial sums should tend toward a finite limit. We conclude that the series converges.

14a) Consider the sequence $a_n = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots$ *Show that* $\lim_{n\to\infty} a_n = 0$*, but* $\sum_{n=1}^{\infty} a_n$ *diverges. Hint:* $s_{10} = 1+(\frac{1}{2}+\frac{1}{2})+(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})+(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4})$

$\lim_{n\to\infty} a_n = 0$, since the fractions involved are getting smaller and smaller.
However, s_n diverges: If you group the terms as above in parentheses, you will notice that the sum inside each set of parentheses is 1. No matter how big s_n gets, there will always be another 1 to add to it if we go far enough. we conclude that the sum has no upper bound, and must diverge to infinity.

14b) Based on your knowledge of improper integrals and your experience in (12), can you think of another (simpler) series that might diverge even though the terms go to zero?

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. To prove this fact, OVERestiate $\frac{1}{x}$ with rectangles. This will show that $\sum_{n=1}^{\infty}\frac{1}{n} > \int_1^{\infty}\frac{1}{x}dx$, which diverges to infinity. It follows that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges as well.